

SEMI-PARALLEL SYMMETRIC OPERATORS FOR HOPF HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS

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ABSTRACT. In this paper, we introduce new notions of semi-parallel shape operators and structure Jacobi operators in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. By using such a semi-parallel condition, we give a complete classification of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$.

INTRODUCTION

The classification of real hypersurfaces in Hermitian symmetric space is one of interesting parts in the field of differential geometry. Among them, we introduce a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ defined by the set of all complex two-dimensional linear subspaces in \mathbb{C}^{m+2} . It is a kind of Hermitian symmetric space of compact irreducible type with rank 2. Remarkably, the manifolds are equipped with both a Kähler structure J and a quaternionic Kähler structure \mathfrak{J} satisfying $JJ_\nu = J_\nu J$ ($\nu = 1, 2, 3$) where J_ν is an orthonormal basis of \mathfrak{J} . When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight. When $m = 2$, we note that the isomorphism $\text{Spin}(6) \simeq \text{SU}(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces in \mathbb{R}^6 . In this paper, we assume $m \geq 3$. (see Berndt and Suh [2] and [3]).

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ and N a local unit normal vector field of M . Since $G_2(\mathbb{C}^{m+2})$ has the Kähler structure J , we may define a *Reeb vector field* ξ defined by $\xi = -JN$ and a 1-dimensional distribution $[\xi] = \text{Span}\{\xi\}$. The Reeb vector field ξ is said to be a *Hopf* if it is invariant under the shape operator A of M . The 1-dimensional foliation of M by the integral curves of ξ is said to be a *Hopf foliation* of M . We say that M is a *Hopf hypersurface* if and if the Hopf foliation of M is totally geodesic. By the formulas in [10, Section 2], it can be easily checked that ξ is Hopf if and only if M is Hopf.

From the quaternionic Kähler structure \mathfrak{J} of $G_2(\mathbb{C}^{m+2})$, there naturally exists *almost contact 3-structure* vector field ξ_1, ξ_2, ξ_3 defined by $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$.

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Put $\mathcal{Q}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$, which is a 3-dimensional distribution in a tangent vector space $T_x M$ of M at $x \in M$. In addition, \mathcal{Q} stands for the orthogonal complement of \mathcal{Q}^\perp in $T_x M$. It becomes the quaternionic maximal subbundle of $T_x M$. Thus the tangent space of M consists of the direct sum of \mathcal{Q} and \mathcal{Q}^\perp as follows: $T_x M = \mathcal{Q} \oplus \mathcal{Q}^\perp$.

For two distributions $[\xi]$ and \mathcal{Q}^\perp defined above, we may consider two natural invariant geometric properties under the shape operator A of M , that is, $A[\xi] \subset [\xi]$ and $A\mathcal{Q}^\perp \subset \mathcal{Q}^\perp$. By using the result of Alekseevskii [1], Berndt and Suh [2] have classified all real hypersurfaces with two natural invariant properties in $G_2(\mathbb{C}^{m+2})$ as follows:

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and \mathcal{Q}^\perp are invariant under the shape operator of M if and only if*

- (A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*
- (B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

In the case (A), we call M is a real hypersurface of Type (A) in $G_2(\mathbb{C}^{m+2})$. Similarly in the case (B) we call M one of Type (B). Using Theorem A, many geometricians have given some characterizations for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with geometric quantities, for example, shape operator, normal (or structure) Jacobi operator, Ricci tensor, and so on. In particular, Lee and Suh [10] gave a characterization for real hypersurfaces of Type (B) as follows:

Theorem B. *Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb vector field ξ belongs to the distribution \mathcal{Q} if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, $m = 2n$, where the distribution \mathcal{Q} denotes the orthogonal complement of \mathcal{Q}^\perp in $T_x M$, $x \in M$. In other words, M is locally congruent to real hypersurfaces of Type (B).*

On the other hand, regarding the parallelism of tensor field T of type $(1,1)$, that is, $\nabla T = 0$, on M in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, there are many well-known results. Among them, when $T = A$ where A denotes the shape operator of M , some geometricians have verified non-existence properties and some characterizations for the shape operator A with many kinds of parallelisms, such as Levi-civita parallel, \mathfrak{F} -parallel, \mathcal{Q}^\perp -parallel, Reeb parallel or generalized Tanaka-Webster parallel, and so on (see [5], [8], [14], [15], etc.).

Furthermore, many geometricians considered such a parallelism for another tensor field of type $(1,1)$ on M , namely, the Jacobi operator R_X defined $(R_X(Y))(p) = (R(Y, X)X)(p)$, where R denotes a Riemannian curvature tensor of type $(1,3)$ on M and X, Y denote tangent vector fields on M . Clearly, each tangent vector field X to M provides the Jacobi operator R_X with respect to X . When it comes to $X = \xi$, the Jacobi operator R_ξ is said to be a *structure Jacobi operator*. Related to the tensor field R_ξ of type $(1,1)$ on M , Pérez, Jeong, and Suh [6] considered the parallelism, that is, $\nabla_X R_\xi = 0$ for any $X \in TM$ and obtained a non-existence property.

In this paper we consider a generalized notion for parallelism of tensor field of type $(1,1)$ on M in $G_2(\mathbb{C}^{m+2})$, namely, semi-parallelism. Actually, in [4] a tensor

field F of type $(1, s)$ on a Riemannian manifold is said to be *semi parallel* if $R \cdot F = 0$. It means that the Riemannian curvature tensor R of M acts as a derivation on F . From this, it is natural that if a tensor field T of type $(1,1)$ is parallel, then T is said to be a *semi-parallel*. Geometricians have proved various results concerning the semi-parallelism conditions of real hypersurfaces in complex space form (see [4], [11], [13]). Recently, K. Panagiotidou and M.M. Tripathi suggested the notion of *semi-parallel normal Jacobi operator* for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (see [12]).

Motivated by these works, we consider semi-parallelisms of the shape operator and the structure Jacobi operator for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, and assert the following theorems, respectively:

Theorem 1. *Let M be a connected real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. There does not exist Hopf hypersurfaces M with semi-parallel shape operator if the smooth function $\alpha = g(A\xi, \xi)$ is constant along the direction of ξ .*

Theorem 2. *Let M be a connected real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. There does not exist Hopf hypersurfaces M with semi-parallel structure Jacobi operator if the smooth function $\alpha = g(A\xi, \xi)$ is constant along the direction of ξ .*

In [12], K. Panagiotidou and M.M. Tripathi proved the following

Theorem C. *There does not exist any connected Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with semi-parallel normal Jacobi operator if the smooth function $\alpha = g(A\xi, \xi) \neq 0$ and \mathcal{Q} - or \mathcal{Q}^\perp -component of ξ is invariant by the shape operator.*

From this we consider that M has a vanishing geodesic Reeb flow when it comes to normal Jacobi operator. Hence by virtue of [9, Lemma 3.1], it gives us a extended result with respect to Theorem C as follows.

Theorem 3. *Let M be a connected real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. There does not exist Hopf hypersurfaces M with normal Jacobi operator if the smooth function $\alpha = g(A\xi, \xi)$ is constant along the direction of ξ .*

In this paper, we refer [1], [2], [3], [10] and [7], [14], [15] for Riemannian geometric structures of $G_2(\mathbb{C}^{m+2})$ and its geometric quantities, respectively.

1. SEMI-PARALLEL SHAPE OPERATOR

In this section, let M represent a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, and R denote the Riemannian curvature tensor of M . Hereafter unless otherwise stated, we consider that X, Y , and Z are any tangent vector field on M . Let W be any tangent vector field on \mathcal{Q} .

We first give the fundamental equation for the semi-parallelism of a tensor field T of type $(1,1)$ on M and prove our Theorem 1.

As mentioned in the introduction, a tensor field T on M is said to be semi-parallel, if T satisfies $R \cdot T = 0$. It is equal to

$$(\dagger) \quad (R(X, Y)T)Z = 0.$$

Since $(R(X, Y)T)Z = R(X, Y)(TZ) - T(R(X, Y)Z)$, the equation (\dagger) is equivalent to the following

$$(\ddagger) \quad R(X, Y)(TZ) = T(R(X, Y)Z).$$

Using this discussion, let us prove our Theorem 1 given in Introduction. In order to do this, suppose that M has the semi-parallel shape operator, that is, the shape operator A of M satisfies the condition $(R(X, Y)A)Z = 0$. From the relation between (\dagger) and (\ddagger) , we see that the given condition is equivalent to

$$(1.1) \quad R(X, Y)(AZ) = A(R(X, Y)Z).$$

Therefore from [14, The equation of Gauss], it becomes

$$\begin{aligned}
 & g(Y, AZ)X - g(X, AZ)Y + g(\phi Y, AZ)\phi X - g(\phi X, AZ)\phi Y \\
 & - 2g(\phi X, Y)\phi AZ + g(AY, AZ)AX - g(AX, AZ)AY \\
 & + \sum_{\nu} \left\{ g(\phi_{\nu}Y, AZ)\phi_{\nu}X - g(\phi_{\nu}X, AZ)\phi_{\nu}Y - 2g(\phi_{\nu}X, Y)\phi_{\nu}AZ \right\} \\
 & + \sum_{\nu} \left\{ g(\phi_{\nu}\phi Y, AZ)\phi_{\nu}\phi X - g(\phi_{\nu}\phi X, AZ)\phi_{\nu}\phi Y \right\} \\
 & - \sum_{\nu} \left\{ \eta(Y)\eta_{\nu}(AZ)\phi_{\nu}\phi X - \eta(X)\eta_{\nu}(AZ)\phi_{\nu}\phi Y \right\} \\
 & - \sum_{\nu} \left\{ \eta(X)g(\phi_{\nu}\phi Y, AZ) - \eta(Y)g(\phi_{\nu}\phi X, AZ) \right\} \xi_{\nu} \\
 (1.2) \quad & = g(Y, Z)AX - g(X, Z)AY + g(\phi Y, Z)A\phi X - g(\phi X, Z)A\phi Y \\
 & - 2g(\phi X, Y)A\phi Z + g(AY, Z)A^2X - g(AX, Z)A^2Y \\
 & + \sum_{\nu} \left\{ g(\phi_{\nu}Y, Z)A\phi_{\nu}X - g(\phi_{\nu}X, Z)A\phi_{\nu}Y - 2g(\phi_{\nu}X, Y)A\phi_{\nu}Z \right\} \\
 & + \sum_{\nu} \left\{ g(\phi_{\nu}\phi Y, Z)A\phi_{\nu}\phi X - g(\phi_{\nu}\phi X, Z)A\phi_{\nu}\phi Y \right\} \\
 & - \sum_{\nu} \left\{ \eta(Y)\eta_{\nu}(Z)A\phi_{\nu}\phi X - \eta(X)\eta_{\nu}(Z)A\phi_{\nu}\phi Y \right\} \\
 & - \sum_{\nu} \left\{ \eta(X)g(\phi_{\nu}\phi Y, Z) - \eta(Y)g(\phi_{\nu}\phi X, Z) \right\} A\xi_{\nu},
 \end{aligned}$$

where \sum_{ν} moves from $\nu = 1$ to $\nu = 3$.

Putting $Y = Z = \xi$ and using the condition of Hopf, the equation (1.2) can be reduced to

$$\begin{aligned}
 & AX + \alpha A^2X \\
 (1.3) \quad & - \sum_{\nu} \left\{ (\eta_{\nu}(X) - \eta(X)\eta_{\nu}(\xi))A\xi_{\nu} + 3\eta_{\nu}(\phi X)A\phi_{\nu}\xi + \eta_{\nu}(\xi)A\phi_{\nu}\phi X \right\} \\
 & = \alpha X + \alpha^2 AX
 \end{aligned}$$

$$-\alpha \sum_{\nu} \left\{ (\eta_{\nu}(X) - \eta(X)\eta_{\nu}(\xi))\xi_{\nu} + 3\eta_{\nu}(\phi X)\phi_{\nu}\xi + \eta_{\nu}(\xi)\phi_{\nu}\phi X \right\}.$$

Our first purpose is to show that ξ belongs to either \mathcal{Q} or \mathcal{Q}^{\perp} .

Lemma 1.1. *Let M be a Hopf hypersurface with semi-parallel shape operator in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If the principal curvature $\alpha = g(A\xi, \xi)$ is constant along the direction of Reeb vector field ξ , then ξ belongs to either the distribution \mathcal{Q} or the distribution \mathcal{Q}^{\perp} .*

Proof. We consider that ξ satisfies

$$(*) \quad \xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$$

for some unit vectors $X_0 \in \mathcal{Q}$, $\xi_1 \in \mathcal{Q}^{\perp}$, and $\eta(X_0)\eta(\xi_1) \neq 0$.

By virtue of [7, Equation (2.10)] and the assumption of $\xi\alpha = 0$, we get $AX_0 = \alpha X_0$ and $A\xi_1 = \alpha\xi_1$.

In the case of $\alpha = 0$, using the equation in [2, Lemma 1],

$$(1.4) \quad Y\alpha = (\xi\alpha)\eta(Y) - 4 \sum_{\nu=1}^3 \eta_{\nu}(\xi)\eta_{\nu}(\phi Y),$$

we obtain that ξ belongs to either \mathcal{Q} or \mathcal{Q}^{\perp} . We next consider the case $\alpha \neq 0$.

We next consider the case $\alpha \neq 0$.

Substituting $X = \phi X_0$ in (1.3) and using basic formulas including (*), we get

$$(1.5) \quad \begin{aligned} & A\phi X_0 - 3\eta(X_0)\eta_1(\xi)A\phi_1\xi + \eta_1(\xi)A\phi_1X_0 - \eta_1(\xi)\eta(X_0)A\phi_1\xi + \alpha A^2\phi X_0 \\ &= \alpha\phi X_0 - 3\alpha\eta(X_0)\eta_1(\xi)\phi_1\xi + \alpha\eta_1(\xi)\phi_1X_0 - \alpha\eta_1(\xi)\eta(X_0)\phi_1\xi + \alpha^2 A\phi X_0. \end{aligned}$$

From (*) and $\phi\xi = 0$, we obtain that $\phi_1\xi = \eta(X_0)\phi_1X_0$ and $\phi X_0 = -\eta(\xi_1)\phi_1X_0$. In addition, substituting X by X_0 into [7, Lemma 2.2] and applying $AX_0 = \alpha X_0$, we see that both vector fields ϕX_0 and ϕ_1X_0 are principal with same corresponding principal curvature $k = \frac{\alpha^2 + 4\eta^2(X_0)}{\alpha}$. From this, (1.5) gives

$$-4k\eta^2(X_0)\phi X_0 + \alpha k^2\phi X_0 - 4\alpha\eta^2(X_0)\phi X_0 - \alpha^2 k\phi X_0 = 0.$$

Since $\alpha \neq 0$, multiplying α to this equation, we obtain

$$4\eta^2(X_0)(8\eta^2(X_0) + \alpha^2)\phi X_0 = 0.$$

By our assumptions, we get $\eta(X_0)\eta(\xi_1) \neq 0$ which means $\phi X_0 = 0$. This makes a contradiction. Accordingly, we get a complete proof of our Lemma. \square

From Lemma 1.1, we only have two cases, $\xi \in \mathcal{Q}$ or $\xi \in \mathcal{Q}^{\perp}$, under our assumptions. Next we further study the case $\xi \in \mathcal{Q}^{\perp}$.

Lemma 1.2. *Let M be a Hopf hypersurface with semi-parallel shape operator in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If the Reeb vector field ξ belongs to the distribution \mathcal{Q}^{\perp} , then M must be a \mathcal{Q}^{\perp} -invariant hypersurface.*

Proof. Since $\xi \in \mathcal{Q}^{\perp}$, we may put $\xi = \xi_1 \in \mathcal{Q}^{\perp}$ for the sake of our convenience. Differentiating $\xi = \xi_1$ along any direction $X \in TM$ and using fundamental formulae in [10, Section 2], it gives us

$$(1.6) \quad \phi AX = 2\eta_3(AX)\xi_2 - 2\eta_2(AX)\xi_3 + \phi_1 AX.$$

Taking the inner product of (1.6) with $W \in \mathcal{Q}$ and taking symmetric part, we also have

$$(1.7) \quad A\phi W = A\phi_1 W.$$

Putting $X = \xi_2$ and $X = \xi_3$ into (1.3), we get, respectively,

$$\begin{cases} 2A\xi_2 + \alpha A^2\xi_2 = 2\alpha\xi_2 + \alpha^2 A\xi_2, \\ 2A\xi_3 + \alpha A^2\xi_3 = 2\alpha\xi_3 + \alpha^2 A\xi_3. \end{cases}$$

For $\alpha = 0$, clearly \mathcal{Q}^\perp is invariant under the shape operator, i.e., $A\mathcal{Q}^\perp \subset \mathcal{Q}^\perp$. Thus, let us consider $\alpha \neq 0$. Then the previous equations imply that

$$(1.8) \quad \begin{cases} A^2\xi_2 = \frac{\alpha^2 - 2}{\alpha} A\xi_2 + 2\xi_2, \\ A^2\xi_3 = \frac{\alpha^2 - 2}{\alpha} A\xi_3 + 2\xi_3. \end{cases}$$

Moreover, restricting $X = \xi_2$, $Y = \xi_3$ and putting $Z = W \in \mathcal{Q}$, the equation (1.2) becomes

$$\begin{aligned} & 4\eta_3(AW)\xi_2 - 4\eta_2(AW)\xi_3 + 2\phi AW - 2\phi_1 AW + \eta_3(A^2W)A\xi_2 - \eta_2(A^2W)A\xi_3 \\ & = 2A\phi W - 2A\phi_1 W + \eta_3(AW)A^2\xi_2 - \eta_2(AW)A^2\xi_3. \end{aligned}$$

Applying (1.6), (1.7) and (1.8) to this equation, it follows $\eta_3(AW)\xi_2 = \eta_2(AW)\xi_3$. This means $\eta_3(AW) = \eta_2(AW) = 0$ for any tangent $W \in \mathcal{Q}$. It completes the proof. \square

From this lemma, we see that M satisfying the assumptions in Lemma 1.2 is locally congruent to a model space of Type (A) in $G_2(\mathbb{C}^{m+2})$. Now, if we assume $\xi \in \mathcal{Q}$, then M with semi-parallel shape operator is locally congruent to one of Type (B) by virtue of Theorem B.

Summing up these discussions, we conclude: *let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If M satisfies (1.1) and $\xi\alpha = 0$, then M must be a model space of Type (A) or (B).*

Hereafter, let us check whether the shape operator of a model space of Type (A) (or one of Type (B)) satisfies the semi-parallel condition (1.1) by [2, Proposition 3] (or [2, Proposition 2], respectively).

Let M_A be a model space of Type (A) in $G_2(\mathbb{C}^{m+2})$. To show our purpose, we suppose that M_A has the semi-parallel shape operator. From (1.3), [2, Proposition 3], and $\xi \in \mathcal{Q}^\perp$, we have

$$(\lambda - \alpha)(2 + \alpha\lambda)X = 0$$

for any tangent vector $X \in T_\lambda = \{X \in T_x M \mid X \perp \xi_\nu, \phi X = \phi_1 X, x \in M\}$. Since $\alpha = \sqrt{8} \cot \sqrt{8}r$ and $\lambda = -\sqrt{2} \tan \sqrt{2}r$ where $r \in (0, \pi/\sqrt{8})$, it implies that every $X \in T_\lambda$ is a zero vector. This gives rise to a contradiction. In fact, the dimension of the eigenspace T_λ is $2m - 2$ where $m \geq 3$.

Now let us consider our problem for a model space of Type (B) denoted by M_B . Similarly, we assume that the shape operator of M_B is semi-parallel. By virtue of [2, Proposition 2], we see that ξ of M_B belongs to \mathcal{Q} . Therefore we obtain $\alpha\beta(\alpha - \beta)\xi_1 = 0$, if we put X as a unit vector field $\xi_1 \in T_\beta$ into (1.3). As we know $\alpha = -2 \tan(2r)$, $\beta = 2 \cot(2r)$ where $r \in (0, \pi/4)$ on M_B , we get a contradiction. This completes the proof of our Theorem 1.

Therefore we assert:

Remark 1.3. The shape operator A of a model space of Type (A) nor Type (B) in $G_2(\mathbb{C}^{m+2})$ does not satisfy the semi-parallelism condition.

Summing up these discussions, we complete the proof of our Theorem 1 given in the introduction. \square

2. SEMI-PARALLEL STRUCTURE JACOBI OPERATOR

In this section, we give a complete prove our Theorem 2. Suppose the structure Jacobi operator of M has semi-parallelism, that is, M satisfies the condition $(R(X, Y)R_\xi)Z = 0$. Besides, from the relation between (\dagger) and (\ddagger) we see that the given condition is equivalent to

$$(2.1) \quad R(X, Y)(R_\xi Z) = R_\xi(R(X, Y)Z).$$

The structure Jacobi operator R_ξ is defined by $R_\xi(X) = R(X, \xi)\xi$, where R denotes the Riemannian curvature tensor on M . Then from the Gauss equation, it can be written as

$$(2.2) \quad \begin{aligned} R_\xi X &= X - \eta(X)\xi + \eta(A\xi)AX - \eta(AX)A\xi \\ &\quad - \sum_{\nu} \left\{ (\eta_\nu(X) - \eta(X)\eta_\nu(\xi))\xi_\nu + 3\eta_\nu(\phi X)\phi_\nu\xi + \eta_\nu(\xi)\phi_\nu\phi X \right\}, \end{aligned}$$

where \sum_{ν} denotes from $\nu = 1$ to $\nu = 3$. From this, we see that $R_\xi\xi = 0$.

Put $Y = Z = \xi$ into (2.1), due to $R_\xi\xi = 0$, we get:

$$(2.3) \quad R_\xi(R_\xi X) = 0.$$

Using these observation from now on we show that ξ belongs to either \mathcal{Q} or its orthogonal complement \mathcal{Q}^\perp such that $TM = \mathcal{Q} \oplus \mathcal{Q}^\perp$.

Lemma 2.1. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with semi-parallel structure Jacobi operator. If the principal curvature $\alpha = g(A\xi, \xi)$ is constant along the direction of ξ , then ξ belongs to either the distribution \mathcal{Q} or the distribution \mathcal{Q}^\perp .*

Proof. Put ξ satisfies $(*)$ for some unit vectors $X_0 \in \mathcal{Q}$ and $\xi_1 \in \mathcal{Q}^\perp$.

Substituting $X = \xi_1$ in (2.2), we have $R_\xi(\xi_1) = \alpha^2\xi_1 - \alpha^2\eta(\xi_1)\xi$. This gives that

$$\begin{aligned} R_\xi(R_\xi\xi_1) &= R_\xi(\alpha^2\xi_1 - \alpha^2\eta(\xi_1)\xi) \\ &= \alpha^2R_\xi\xi_1 - \alpha^2\eta(\xi_1)R_\xi\xi \\ &= \alpha^4\xi_1 - \alpha^4\eta(\xi_1)\xi. \end{aligned}$$

So, the condition of semi-parallel structure Jacobi operator implies

$$\alpha^4\xi_1 - \alpha^4\eta(\xi_1)\xi = 0.$$

From this, taking the inner product with $X_0 \in \mathcal{Q}$, it gives $\alpha^4\eta(\xi_1)\eta(X_0) = 0$. So we obtain the following three cases: $\alpha = 0$, $\eta(X_0) = 0$ or $\eta(\xi_1) = 0$. When α is identically vanishing, by virtue of (1.4) we conclude that ξ belongs to either \mathcal{Q} or \mathcal{Q}^\perp . For $\eta(\xi_1) = 0$, then ξ belongs to \mathcal{Q} because of our notation $(*)$. Moreover, ξ belongs to \mathcal{Q}^\perp if $\eta(X_0) = 0$. Accordingly, it completes the proof of our Lemma. \square

According to Lemma 2.1, we consider the case $\xi \in \mathcal{Q}^\perp$.

Lemma 2.2. *Let M be a Hopf hypersurface with semi-parallel structure Jacobi operator in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If the Reeb vector field ξ belongs to the distribution \mathcal{Q}^\perp , then $g(A\mathcal{Q}, \mathcal{Q}^\perp) = 0$.*

Proof. We may put $\xi = \xi_1$, because $\xi \in \mathcal{Q}^\perp$. Differentiating $\xi = \xi_1$ for any direction X on M , we obtain

$$(2.4) \quad \begin{cases} q_2(X) = 2g(AX, \xi_2), \quad q_3(X) = 2g(AX, \xi_3) \text{ and} \\ AX = \eta(AX)\xi + 2g(AX, \xi_2)\xi_2 + 2g(AX, \xi_3)\xi_3 - \phi\phi_1AX \\ \quad (\text{or } AX = \eta(X)A\xi + 2\eta_2(X)A\xi_2 + 2\eta_3(X)A\xi_3 - A\phi\phi_1X). \end{cases}$$

Putting $X = \xi_2$ into (2.2), it follows that $R_\xi(\xi_2) = 2\xi_2 + \alpha A\xi_2$. If the smooth function α vanishes, it makes a contradiction. In fact, from (2.3) we see that $R_\xi(R_\xi\xi_2) = 4\xi_2 = 0$. Thus we may consider that the smooth function α is non-vanishing.

On the other hand, it follows that for any $W \in \mathcal{Q}$ the equation (2.2) becomes

$$R_\xi(W) = W + \phi_1\phi W + \alpha AW,$$

from this, together with the semi-parallelism of R_ξ , it follows that

$$(2.5) \quad \begin{aligned} 0 &= R_\xi(R_\xi W) \\ &= 2\alpha AW + 2\alpha\eta_3(AW)\xi_3 + 2\alpha\eta_2(AW)\xi_2 - \alpha\phi_1\phi AW \\ &\quad + \alpha^2 A^2 W + \alpha A\phi_1\phi W. \end{aligned}$$

From (2.4) and $\alpha \neq 0$, it follows that $2AW + \alpha A^2 W = 0$, where $AW = -A\phi_1\phi W$ for any tangent vector field $W \in \mathcal{Q}$. Taking the inner product with ξ_2 and ξ_3 , respectively, it becomes

$$(2.6) \quad \alpha\eta_2(A^2 W) = -2\eta_2(AW), \quad \alpha\eta_3(A^2 W) = -2\eta_3(AW).$$

Moreover, according to (2.2), we also have $R_\xi(A\xi_2) = 2A\xi_2 + \alpha A^2\xi_2$, which induces that

$$\begin{aligned} 0 &= R_\xi(R_\xi\xi_2) = R_\xi(2\xi_2 + \alpha A\xi_2) \\ &= 2R_\xi(\xi_2) + \alpha R_\xi(A\xi_2) \\ &= 4\xi_2 + 4\alpha A\xi_2 + \alpha^2 A^2\xi_2. \end{aligned}$$

Again taking the inner product with $W \in \mathcal{Q}$ and using the fact $\alpha \neq 0$, we have

$$(2.7) \quad \alpha\eta_2(A^2 W) = -4\eta_2(AW).$$

From this and (2.6), we obtain $\eta_2(AW) = 0$ for any tangent vector field $W \in \mathcal{Q}$.

Similarly, from (2.2) we get $R_\xi\xi_3 = 2\xi_3 + \alpha A\xi_3$ and $R_\xi(A\xi_3) = 2A\xi_3 + \alpha A^2\xi_3$, which gives

$$(2.8) \quad \begin{aligned} 0 &= R_\xi(R_\xi\xi_3) = R_\xi(2\xi_3 + \alpha A\xi_3) \\ &= 4\xi_3 + 4\alpha A\xi_3 + \alpha^2 A^2\xi_3. \end{aligned}$$

From this, taking the inner product with $W \in \mathcal{Q}$ and using $\alpha \neq 0$, we have $4\eta_3(AW) + \alpha\eta_3(A^2 W) = 0$. Combining this and (2.6), we get also $\eta_3(AW) = 0$ for any $W \in \mathcal{Q}$. Until now, we have proven if M satisfies our assumptions, then the distribution \mathcal{Q}^\perp is invariant under the shape operator, that is, $g(A\mathcal{Q}, \mathcal{Q}^\perp) = 0$. This gives a complete proof of our lemma. \square

From this lemma and Theorem A given by Berndt and Suh [2], we see that a Hopf hypersurface M satisfying the assumptions in Lemma 2.2 is locally congruent to a model space of Type (A). Now, if ξ belongs to \mathcal{Q} , then by virtue of Theorem B a Hopf hypersurface M with semi-parallel structure Jacobi operator is locally congruent to a real hypersurface of Type (B) in $G_2(\mathbb{C}^{m+2})$. Hence we conclude that let

M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. If M satisfies (2.1) and $\xi\alpha = 0$, then M is a model space of Type (A) or (B).

From such a point of view, let us consider the converse problem. More precisely, we check whether the structure Jacobi operator R_ξ of a model space of Type (A) (or of Type (B), resp.) satisfies the semi-parallel condition (2.1).

In order to check our problem for a model space M_A , we suppose that M_A has the semi-parallel structure Jacobi operator. By virtue of Proposition 3 in [2], we see that $\xi = \xi_1 \in T_\alpha$ and $\xi_j \in T_\beta$ for $j = 2, 3$. From this, the semi-parallel condition for R_ξ becomes

$$\begin{aligned} R_\xi(R_\xi\xi_2) &= 4\xi_2 + 4\alpha\beta\xi_2 + \alpha^2\beta^2\xi_2 \\ &= (\alpha\beta + 2)^2\xi_2 = 0 \end{aligned}$$

when we put $X = \xi_2$ in (2.3). It implies $(\alpha\beta + 2) = 0$. But since $\alpha = \sqrt{8}\cot(\sqrt{8}r)$ and $\beta = \sqrt{2}\cot(\sqrt{2}r)$, we obtain $(\alpha\beta + 2) = 2\cot^2(\sqrt{2}r) \neq 0$ for $r \in (0, \pi/2\sqrt{2})$. Thus it gives us a contradiction.

In the sequel, we check whether R_ξ of a model space M_B of Type (B) is semi-parallel. To do this, we assume that R_ξ of M_B satisfies the condition (2.1). On a tangent vector space $T_x M_B$ at any point $x \in M_B$, the Reeb vector ξ belongs to \mathcal{Q} . From this and (2.2), the condition of (2.1) implies that for $X = \xi_2 \in T_\beta$

$$R_\xi(R_\xi\xi_2) = \alpha^2\beta^2\xi_2 = 0.$$

On the other hand, from [2, Proposition 2], since $\alpha = -2\tan(2r)$ and $\beta = 2\cot(2r)$ where $r \in (0, \pi/4)$ on M_B , we get $(\alpha\beta)^2 = 16$. So, we consequently see that the tangent vector ξ_2 must be zero, which gives a contradiction.

Therefore we assert:

Remark 2.3. The structure Jacobi operator R_ξ of a model space of Type (A) nor Type (B) in $G_2(\mathbb{C}^{m+2})$ does not satisfy the semi-parallelism condition.

Summing up these discussions, we complete the proof of our Theorem 2 given in the introduction. \square

3. SEMI-PARALLEL NORMAL JACOBI OPERATOR

Now, we observe a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with semi-parallel normal Jacobi operator, that is, the normal Jacobi operator \bar{R}_N of M satisfies

$$(R(X, Y)\bar{R}_N)Z = 0$$

for all tangent vector fields X, Y, Z on M .

In order to prove Theorem 3 mentioned in Introduction, let us consider the case that M has vanishing geodesic Reeb flow.

Lemma 3.1. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with vanishing geodesic Reeb flow. If the normal Jacobi operator \bar{R}_N of M is semi-parallel, then M is locally congruent to a model space of Type (A) or Type (B).*

Proof. When the function $\alpha = g(A\xi, \xi)$ identically vanishes, it can be seen directly by (1.4) that ξ can be divided into $\xi \in \mathcal{Q}$ or $\xi \in \mathcal{Q}^\perp$. Then we first consider the case

that ξ belongs to \mathcal{Q} . By virtue of Theorem B, we get that M is locally congruent to a model space of Type (B).

Next, we consider the case $\xi \in \mathcal{Q}^\perp$. Substitution of the previous two relations in [12, (4.17)] gives

$$(3.1) \quad \begin{aligned} & 7W + 7\alpha AW - 6\phi_1\phi W \\ & = 2\alpha\eta_2(AW)\xi_2 + 2\alpha\eta_3(AW)\xi_3 + \phi_1\phi(\phi_1\phi W) - \alpha\phi_1\phi AW. \end{aligned}$$

Since $\alpha = 0$, it follows that $7W - 6\phi_1\phi W = \phi_1\phi(\phi_1\phi W)$ for any $W \in \mathcal{Q}$. Moreover, from $\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu$, $\nu = 1, 2, 3$, we obtain $\phi\phi_1(\phi\phi_1 W) = W$. Thus (3.1) implies $\phi_1\phi W = W$. It implies $AW = 0$ for any $W \in \mathcal{Q}$, together with (2.4). It gives us a complete proof for $\alpha = 0$. \square

It remains to be checked if the normal Jacobi operator \bar{R}_N of a model space M_A or M_B satisfy the semi-parallelism condition. For $\xi \in \mathcal{Q}^\perp$, we easily get $2\xi = 0$ from [12, Equations (5.2) and (5.3)]. For $\xi \in \mathcal{Q}$, as we know $\alpha = -2\tan(2r)$ with $r \in (0, \pi/4)$ on a real hypersurface of Type (B), α never vanishes (see [2, Proposition 2]). So, neither the normal Jacobi operator \bar{R}_N of M_A nor M_B does not satisfy the semi-parallelism condition. Thus we get the following:

Corollary 3.2. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with vanishing geodesic Reeb flow. Then there does not exist any Hopf hypersurface if the normal Jacobi operator \bar{R}_N of M satisfies the condition of semi-parallelism.*

Combining Theorem C and Corollary 3.2, we give a complete proof of Theorem 3 in the introduction. \square

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